DIGIT SUMS, PERIODIC CONGUENCE, AND THE PRIME AND PENTAGONAL NUMBERS

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Definition 1 (Digit Sums of all $N \in \mathbb{Z}$)**.** Taking any particular radix R, we observe only $R-1$ unique digit sum cycles per instance R; a digit sum itself being merely the adding of any integer's digits until a value less than R is reached. In this sense, the concept of congruence (indicated by $\equiv$) as well as residue class modulated with respect to a given integer, and digit sums are related. In the same way that division can be said to presuppose periodic congruences (ratios of divisibility) generic arithmetic and algebraic operations per radix R may be said to presuppose cyclic digit sums $ds_{\{Z\}}N$ for all multiples of any integers N. [a]

$$ds_{\{Z\}}N = \prod_{j=m}^{m\to\infty} X_j Y_k = \pi_m \equiv \pi_{m\pm x_i} = N_{k'} \equiv N_{k'\pm(R-1)} \pmod{R-1} = \{r_m\}.$$

The operator ds is delimited by: $Z = \{z|0 \leq z \leq R-1 \in \mathbb{Z}\}$, the set of coefficients less than some radix R; $X = \{x|x \in \mathbb{Z}\}$; the index m as which multiple of $R-1$ i.e. $\{m|m = N_i := \prod_{i=0}^{i\to\infty} \mathbb{Z}_i(R-1)\}$; $\{Y\} := \{Z\}$; the index k denoting which integer coefficient Y (less than R) of X_j; π_m as any particular product in the field X_jY_k ; the congruence $\pi_m \equiv \pi_{m\pm x_j} \pmod{R-1}$ shows that for any sequence of products in which only k is caused to vary per given instance j, the difference on the index m of the residue class $r = \{r|0 \leq r \leq R-1\}$ for the products $\prod$ shall be $X_j(R-1)$.

[a] See Table 5, page 7, for one example, in $R = 10$.

Definition 2. Let it be denoted that all primes $\{\mathbb{P}\}$ (excepting 2 and 3) comprise some set, say $\{P\}$ such that within the field of integers

$$\mathbb{Z} \supset \mathbb{T} \supset P \wedge T.$$

Some infinite series $\mathbb{T}$ contains some other series P as well as some series T which contains infinitely many series T' derived from T by some operation, to be established hereafter. K generally as an index shall denote any k-th term is simply some $k = n \in \mathbb{Z}$, such that we may distinguish any k-th term in $\mathbb{T}$, any k-th series T' contained in $\mathbb{T}$, etc.

Definition 3 (Constraints on set P)**.** Within the set of primes, the numbers 2, 3 and 5 are exceptional. Obviously 2 is the only even prime; multiples of 3 always have a digit sum of 3, 6, or 9 after successive repettition; multiples of 5 obviously end in a character (within $R = 10$) denoting either zero (a placeholer indicating proportionality to the base) or else five itself. Obviously, $2 + 3 = 5$.

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Now, given the vector

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix} \begin{bmatrix} +2 & +4 & +2 & +4 & \cdots \\ +2 & +4 & -10 & +4 & \end{bmatrix}$$

which begins at the square with sides negative one on the $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ plane, successively adding, then subtracting in the oscillating manner defined by any four successive columns in our matrix; defining some initial 0-th term $\mu = -1$ such that

$$\sum_{k=-\infty}^{k=+\infty} \mu_k = \{\cdots -2_{-2} -4_{-1} -1_0 +2_1 +4_2 +2_3 \cdots\} = \{N_{k>0} \equiv N_{k+4}\} \pmod{12} = \{\mathbb{T}\};$$

given the conditions (for all $k \neq 0$):

$$\mu_k = \begin{cases} -2 & if\ k < 0 \equiv 0 \\ -4 & if\ k < 0 \equiv 1 \\ +2 & if\ k > 0 \equiv 1 \\ +4 & if\ k > 0 \equiv 0 \end{cases} \pmod 2;$$

with some variable t determined by

$$t\,(k) = \frac{k}{2} - 1,$$

such that for any k-th term in $\mathbb{T}$ the expression

$$\mathbb{T}_k = \sum_{k=0}^{i=k} \mu_k = \begin{cases} 3k - \mu_k & if\ k > 0 \equiv 1 \\ 3k - C + t & if\ k > 0 \equiv 0 \end{cases} \pmod 2 \tag{0.1}$$

holds for some constant rate C. Setting x_i equal to some $\mathbb{T}_k$ for all k the conditions

$$x_i = \mathbb{T}_k = \sum_{k=0}^{i=k} \mu_k$$

and

$$y_i = \begin{cases} +1 & if\ k > 0 \equiv 1 \\ +5 & if\ k > 0 \equiv 2 \\ -5 & if\ k > 0 \equiv 3 \\ -1 & if\ k > 0 \equiv 0 \end{cases} \pmod 4$$

hold. $\mathbb{T}_0 = \mu$, therefore it follows that $\mathbb{T}_k + \mathbb{T}_{k+1} = 12x$ for all even $k \geq 0$. The differences between every term $\mathbb{T}_k^2$ become

$$\mathbb{T}_k^2 - \mathbb{T}_{k-1}^2 = 24x \begin{cases} x = \{0, 1, 3, 5, 7, 9, \cdots, \infty\} & if\ k > 0 \equiv 1 \\ x = \{0, 1, 2, 3, 4, 5, \cdots, \infty\} & if\ k > 0 \equiv 0 \end{cases} \pmod 2.$$

One correlation to this fact is that the sum of the differences of the k-values $k_n - k_{n-1}$ of any consecutive squares S contained in $\mathbb{T}$ will be equal to $2^3 C_i$ for the rate C.

$$S_{k'}(\mathbb{T}) = 1 + \sum_{i=1}^{i=n} 2^3 C_i = k'_j \tag{0.2}$$

The above equation yields the transformation of any initial value k of $\mathbb{T}_k$ to the k-value k' for all squares $S_{k'} = \mathbb{T}_k^2$. As such, in accordance with equation (0.1), it folows that $\mathbb{T}_1^2 - \mathbb{T}_0^2 = 0$ is the first value of $2^3 C_i$ and hence, the value $k' = 1$.

Definition 4 (Differences of consecutive Pentagonal Numbers)**.** As noted in *On the Remarkable Properties of the Pentagonal Numbers* and elsewhere by *Euler* (granting $\P_k$ as denoting any k-th such number), generalized pentagonal numbers are of the form $\frac{n(3n\pm1)}{2}$. Subsequently, it may be demonstrated that

$$\P_k = \sum_{i,j=1}^{i,j=n+1} \begin{cases} (3n+1)_i & if\ k>0\equiv 1 \\ (3n+2)_j & if\ k>0\equiv 0 \end{cases} \begin{cases} i=k-n & \iff k>0\equiv 1 \\ j=k/2 & \iff k>1\equiv 0 \end{cases} \pmod 2,$$

$n \geq 0$; the values of the i-th sum correspond to each value of $3n-1$ and those of every j-th sum to each value $3n+1$ (per Euler's original equation). Indexing each term progressing according to magnitude, the constant C may be defined by the difference of any two consecutive terms: $\P_k - \P_{k-1}$. A recurrence relation in the set of differences, $\{\Delta\}_i = \{1_1, 1_2, 3_3, 2_4, 5_5, 3_6, \cdots, \Delta_n\} = \{C\}_i$, follows such that:

$$\Delta_n = 2\Delta_{n-2} - \Delta_{n-4},\ n \geq 5.$$

Incidentally, the set $\mathbb{T}$ outlined earlier conforms to an identical recurrence relation, namely:

$$\mathbb{T}_n = 2\mathbb{T}_{n-2} - \mathbb{T}_{n-4},\ n \geq 5.$$

Further, for any consecutive differences of any k-th $\mathbb{T}$ and $\P$ it holds that

$$(\mathbb{T}_{k+1} - \P_{k+1}) - (\mathbb{T}_k - \P_k) = \begin{cases} -1, -3-5, -7, -9, \cdots & if\ k>0\equiv 1 \\ +3, +2, +1, +0, -1, \cdots & if\ k>0\equiv 0 \end{cases} \pmod 2.$$

1. A proof of the regularity of primes, by finite induction

Lemma (Periodic congruences within P at all powers, for all products)**.** *All members of* $\mathbb{T}$ *conform to a digit sum cycle with period* $\bar{6}$ *with* $9x = m$ *such that for all terms therein it holds that* $\mathbb{T}_{k>0} \equiv \mathbb{T}_{k+6} \pmod 9$. *The cyclic residue is*

$$r_m = \{1_0, 5_0, 7_0, 2_1, 4_1, 8_1, 1_2, \cdots\} \pmod 9.$$

Proof. See Table 1. □

Corollary. *Terms in* $\mathbb{T}$ *occur periodically relative to* $12x = m$*, period* $\bar{4}$*, where* $\mathbb{T}^n_{k>0} \equiv \mathbb{T}^n_{k+4} \pmod{12} = r_{m(\rho)}$ *for some rate* $\rho(z) = \Delta m : n$ *such that*

$$\mathbb{T}^1_k = r_m = \{1_0, 5_0, 7_0, 11_0, 1_1, \cdots\} \pmod{12}.$$

Proof. See Table 2. □

Corollary. *All members of* $\mathbb{T}$ *conform to a periodic repetition of final digit with period* $\bar{10}$ *such that* $\mathbb{T}_{k>0} \equiv \mathbb{T}_{k+10} \pmod{30} = r_m$. *The cyclic modular residue is*

$$r_m = \{1_0, 5_0, 7_0, 11_0, 13_0, 17_0, 19_0, 23_0, 25_0, 29_0, 1_1, \cdots\} \pmod{30}.$$

Proof. See Table 3. □

k	μ	C	t	$\vec{\mu}$	x_i	y_i	T'_1	T'_2	T'_3	T'_4	T'_5	T'_6
1	2	1	-0.5	$3k-\mu$	1_1	$+1$	1_0	5_0	7_0	2_1	4_1	8_1
2	4	1	0	$3k-C+t$	5_2	$+5$	7_2	8_3	1_6	2_7	4_9	5_{10}
3	2	3	0.5	$3k-\mu$	7_3	-5	4_5	5_8	1_{10}	2_{13}	7_{14}	8_{17}
4	4	2	1	$3k-C+t$	11_4	-1	4_{13}	8_{15}	7_{20}	2_{23}	1_{28}	5_{30}
5	2	5	1.5	$3k-\mu$	13_5	$+1$	7_{18}	5_{24}	4_{27}	2_{33}	1_{36}	8_{41}
6	4	3	2	$3k-C+t$	17_6	$+5$	1_{32}	8_{35}	4_{43}	2_{47}	7_{54}	5_{58}

TABLE 1. $\prod_{z\geq 0}^{z\to\infty} x_i x_{i+z} \equiv \prod_{z\geq 0}^{z\to\infty} x_{i+6v} x_{i+z+6w} \pmod 9 = r_m$

Remark. Table 3 (on the following page) contains all terms $\mathbb{T}_k < 600$. The notational shortdand $\binom{d}{k'}^n$ designates $d = x_i = \mathbb{T}_k$ and $k' = x_{i+z}$ at any value, including k of $\mathbb{T}$ at $\mathbb{T}_k$ with $z = 0$. Only terms which are not composite which are in $\mathbb{T}$ are enumerated by unique integer values for some instance k'. For some terms, x_{i+z} is denoted as a composite from some lesser series T' (e.g. $\binom{7}{(5/4)^1}^1 = 5 \times 7 \times 11$). [a] The number 1 (being its own square) recursively yields the set $\mathbb{T}$.

[a]Read as "seven choose (five choose four)"; intended to indicate $\binom{7}{3}^1\binom{5}{4}^1 = 385$.

Conjecture (Sketch of a General Proof)**.** *While the following argument is perhaps incomplete, allowing π_n to denote any iterated product, any series T'_k being delimited by*

$$T'_k = \prod_{i=k}^{n\to\infty} x_i x_{i+n} = \pi_1, \pi_2, \pi_3, \cdots \pi_n \tag{1.1}$$

we may evince that the pattern of differences produced by the equation

$$\Delta T'_k = \pi_n - \pi_{n-1}$$

yields differences of the form

$$\Delta T'_k = \begin{cases} 4\mathbb{T}_k, 2\mathbb{T}_k, 4\mathbb{T}_k, 2\mathbb{T}_k, \cdots & if\ k > 0 \equiv 1 \\ 2\mathbb{T}_k, 4\mathbb{T}_k, 2\mathbb{T}_k, 4\mathbb{T}_k, \cdots & if\ k > 0 \equiv 0 \end{cases} \pmod 2$$

such that for all $\mathbb{T}_k = T'_{k'}$ including all composites $(x_i x_{i+n})$ therein, it holds that

$$\frac{(x_i x_{i+n} + 30x_i)}{x_i} = x_{i+10}. \tag{1.2}$$

The summand is always some term in the same series T'_k. Further, allowing some operator λ of T'_k to cancel out any composite terms in $\mathbb{T}$ which occur in more than one series T' (such as $\binom{7}{(5/4)^1}^1 = 385$), in such case it would then follow that

$$\frac{\sum_{k=1}^{z\to\infty} \mathbb{T}_{k+z} - \sum_{k=2}^{k\to\infty} T'_k}{\lambda_n(T'_k)} - \sum_{\lambda_0}^{\lambda_n} Q_n = P \in \mathbb{T} \tag{1.3}$$

Q_n being the quotient of λ_n. $\mathbb{P}$ as the set of all primes, it follows that [a]

$$5 + \sum_{k=1}^{k\to\infty} P_k = 1 + \sum_{k=1}^{i=k+1} \mathbb{P}_k. \tag{1.4}$$

[a]I.e. accounting for our exceptions on one side $(2 + 3 = 5)$, and the nonprime unit 1 on the other.

P_k^n	$\binom{n=0}{r_m}$	$\binom{n=1}{r_m}$	$\binom{n=2}{r_m}$	$\binom{n=3}{r_m}$	$\binom{n=4}{r_m}$	$\binom{n=5}{r_m}$	$\binom{n=6}{r_m}$	$\binom{n=7}{r_m}$
1_1	1_0	1_0	1_0	1_0	1_0	1_0	1_0	1_0
5_2	1_0	5_0	1_2	5_{10}	1_{52}	5_{260}	1_{1302}	5_{6510}
7_3	1_0	7_0	1_4	7_{28}	1_{200}	7_{1400}	1_{9804}	7_{68628}
11_4	1_0	11_0	1_{10}	11_{110}	1_{1220}	11_{13420}	1_{147630}	$11_{1623930}$
13_5	1_0	1_1	1_{14}	1_{183}	1_{2380}	1_{30941}	1_{402234}	$1_{5229043}$
17_6	1_0	5_1	1_{24}	5_{409}	1_{6960}	5_{118321}	$1_{2011464}$	$5_{34194889}$
19_7	1_0	7_1	1_{30}	7_{571}	1_{10860}	7_{206341}	$1_{3920490}$	$7_{74489311}$
23_8	1_0	11_1	1_{44}	11_{1013}	1_{23320}	11_{536361}	$1_{12336324}$	$11_{283735453}$
25_9	1_0	1_2	1_{52}	1_{1302}	1_{32552}	1_{813802}	$1_{20345052}$	$1_{508626302}$
29_{10}	1_0	5_2	1_{70}	5_{2032}	1_{58940}	$5_{1709262}$	$1_{49568610}$	$5_{1437489692}$
31_{11}	1_0	7_2	1_{80}	7_{2482}	1_{76960}	$7_{2385762}$	$1_{73958640}$	$7_{2292717842}$
35_{12}	1_0	11_2	1_{102}	11_{3572}	1_{125052}	$11_{4376822}$	$1_{15318802}$	$11_{5361608072}$

TABLE 2. $\mathbb{T}^n_{k\geq 1} \equiv \mathbb{T}^n_{k+4} \pmod{12} = r_{m(\rho)}$

Theorem. *All series T'_k including 1_1 (recursively) generate period $\bar{4}$ waveforms on the $\binom{x}{y}$ plane. Definition (0.3) demonstrates a maximum of $3!$ choices for any series T'. In fact, only $2!$ choices exist, such that*

$$f(\vec{k}) = \sum_{k=1}^{i=k} 2^3 C_i + N_{[y_1,y_2,y_3,y_4]} \begin{cases} y_1 \equiv 1 & \forall\, k \\ y_2 \begin{cases} \equiv 5 & if\ k\ is\ odd \\ \equiv -1 & if\ k\ is\ even \end{cases} & \forall\, k \\ y_3 \equiv -5 & \forall\, k \\ y_4 \begin{cases} \equiv 5 & if\ k\ is\ even \\ \equiv -1 & if\ k\ is\ odd \end{cases} & \forall\, k \end{cases} \pmod{12}. \tag{1.5}$$

N is determined by $\mathbb{T}_{k\geq 1} \equiv \mathbb{T}_{k+6} \pmod{18} = r_m$. Any six terms in $\mathbb{T}$ leave periodic residues $\pmod{18}$. Every other sixth term yields an equivalent digit sum.

$$\begin{bmatrix} 1 & 19 & 37 & 55 & 73 & 91 & \cdots & N \equiv 1_m \\ 5 & 23 & 41 & 59 & 77 & 95 & \cdots & N \equiv 5_m \\ 7 & 25 & 43 & 61 & 79 & 97 & \cdots & N \equiv 7_m \\ 11 & 29 & 47 & 65 & 83 & 101 & \cdots & N \equiv 11_m \\ 13 & 31 & 49 & 67 & 85 & 103 & \cdots & N \equiv 13_m \\ 17 & 35 & 53 & 71 & 89 & 107 & \cdots & N \equiv 17_m \end{bmatrix} \pmod{18}.$$

1_m	5_m	7_m	11_m	13_m	17_m	19_m	23_m	25_m	29_m
$\binom{1}{1}^n$	$\binom{5}{2}^1$	$\binom{7}{3}^1$	$\binom{11}{4}^1$	$\binom{13}{5}^1$	$\binom{17}{6}^1$	$\binom{19}{7}^1$	$\binom{23}{8}^1$	$\binom{5}{2}^2$	$\binom{29}{9}^1$
$\binom{31}{10}^1$	$\binom{5}{3}^1$	$\binom{37}{11}^1$	$\binom{41}{12}^1$	$\binom{43}{13}^1$	$\binom{47}{14}^1$	$\binom{7}{3}^2$	$\binom{53}{15}^1$	$\binom{5}{4}^1$	$\binom{59}{16}^1$
$\binom{61}{17}^1$	$\binom{5}{5}^1$	$\binom{67}{18}^1$	$\binom{71}{19}^1$	$\binom{73}{20}^1$	$\binom{7}{4}^1$	$\binom{79}{21}^1$	$\binom{83}{22}^1$	$\binom{5}{6}^1$	$\binom{89}{23}^1$
$\binom{7}{5}^1$	$\binom{5}{7}^1$	$\binom{97}{24}^1$	$\binom{101}{25}^1$	$\binom{103}{26}^1$	$\binom{107}{27}^1$	$\binom{109}{28}^1$	$\binom{113}{29}^1$	$\binom{5}{8}^1$	$\binom{7}{6}^1$
$\binom{11}{4}^2$	$\binom{5}{2}^3$	$\binom{127}{30}^1$	$\binom{131}{31}^1$	$\binom{7}{7}^1$	$\binom{137}{32}^1$	$\binom{139}{33}^1$	$\binom{11}{5}^1$	$\binom{5}{9}^1$	$\binom{149}{34}^1$
$\binom{151}{35}^1$	$\binom{5}{10}^1$	$\binom{157}{36}^1$	$\binom{7}{8}^1$	$\binom{163}{37}^1$	$\binom{167}{38}^1$	$\binom{13}{5}^2$	$\binom{173}{39}^1$	$\binom{7}{(5/2)^2}^1$	$\binom{179}{40}^1$
$\binom{181}{41}^1$	$\binom{5}{11}^1$	$\binom{11}{6}^1$	$\binom{191}{42}^1$	$\binom{193}{43}^1$	$\binom{197}{44}^1$	$\binom{199}{45}^1$	$\binom{7}{9}^1$	$\binom{5}{12}^1$	$\binom{11}{7}^1$
$\binom{211}{46}^1$	$\binom{5}{13}^1$	$\binom{7}{10}^1$	$\binom{13}{6}^1$	$\binom{223}{47}^1$	$\binom{227}{48}^1$	$\binom{229}{49}^1$	$\binom{231}{50}^1$	$\binom{5}{14}^1$	$\binom{239}{51}^1$
$\binom{241}{52}^1$	$\binom{5}{(7/3)^2}^1$	$\binom{13}{7}^1$	$\binom{251}{53}^1$	$\binom{11}{8}^1$	$\binom{257}{54}^1$	$\binom{7}{11}^1$	$\binom{263}{55}^1$	$\binom{5}{15}^1$	$\binom{269}{56}^1$
$\binom{271}{57}^1$	$\binom{11}{(5/2)^2}^1$	$\binom{277}{58}^1$	$\binom{281}{59}^1$	$\binom{283}{60}^1$	$\binom{7}{12}^1$	$\binom{17}{6}^2$	$\binom{293}{61}^1$	$\binom{5}{16}^1$	$\binom{13}{8}^1$
$\binom{7}{13}^1$	$\binom{5}{17}^1$	$\binom{307}{62}^1$	$\binom{311}{63}^1$	$\binom{313}{64}^1$	$\binom{317}{65}^1$	$\binom{11}{29}$	$\binom{17}{7}^1$	$\binom{13}{(5/2)^2}^1$	$\binom{7}{14}^1$
$\binom{331}{66}^1$	$\binom{5}{18}^1$	$\binom{337}{67}^1$	$\binom{11}{10}^1$	$\binom{7}{3}^3$	$\binom{347}{68}^1$	$\binom{349}{69}^1$	$\binom{353}{70}^1$	$\binom{5}{19}^1$	$\binom{359}{71}^1$
$\binom{19}{7}^2$	$\binom{5}{20}^1$	$\binom{367}{72}^1$	$\binom{7}{15}^1$	$\binom{373}{73}^1$	$\binom{13}{9}^1$	$\binom{379}{74}^1$	$\binom{383}{75}^1$	$\binom{7}{(5/4)^1}^1$	$\binom{389}{76}^1$
$\binom{17}{8}^1$	$\binom{5}{21}^1$	$\binom{397}{77}^1$	$\binom{401}{78}^1$	$\binom{13}{10}^1$	$\binom{11}{11}^1$	$\binom{409}{79}^1$	$\binom{7}{16}^1$	$\binom{5}{22}^1$	$\binom{419}{80}^1$
$\binom{421}{81}^1$	$\binom{17}{(5/2)^2}^1$	$\binom{7}{17}^1$	$\binom{431}{82}^1$	$\binom{433}{83}^1$	$\binom{19}{8}^1$	$\binom{439}{84}^1$	$\binom{443}{85}^1$	$\binom{5}{23}^1$	$\binom{449}{86}^1$
$\binom{11}{12}^1$	$\binom{7}{(5/5)^1}^1$	$\binom{457}{87}^1$	$\binom{461}{88}^1$	$\binom{463}{89}^1$	$\binom{467}{90}^1$	$\binom{7}{18}^1$	$\binom{11}{13}^1$	$\binom{19}{(5/2)^2}^1$	$\binom{479}{91}^1$
$\binom{13}{11}^1$	$\binom{5}{24}^1$	$\binom{487}{92}^1$	$\binom{491}{93}^1$	$\binom{17}{9}^1$	$\binom{7}{19}^1$	$\binom{499}{94}^1$	$\binom{503}{95}^1$	$\binom{5}{25}^1$	$\binom{509}{96}^1$
$\binom{7}{20}^1$	$\binom{5}{26}^1$	$\binom{11}{14}^1$	$\binom{521}{97}^1$	$\binom{523}{98}^1$	$\binom{17}{10}^1$	$\binom{23}{8}^2$	$\binom{13}{12}^1$	$\binom{5}{27}^1$	$\binom{11}{3^2}^1$
$\binom{541}{99}^1$	$\binom{5}{28}^1$	$\binom{547}{100}^1$	$\binom{19}{9}^1$	$\binom{7}{21}^1$	$\binom{557}{101}^1$	$\binom{13}{13}^1$	$\binom{563}{102}^1$	$\binom{5}{29}^1$	$\binom{569}{103}^1$
$\binom{571}{104}^1$	$\binom{23}{(5/2)^2}^1$	$\binom{577}{105}^1$	$\binom{7}{22}^1$	$\binom{11}{15}^1$	$\binom{587}{106}^1$	$\binom{19}{10}^1$	$\binom{593}{107}^1$	$\binom{7}{(5/6)^1}^1$	$\binom{599}{108}^1$

TABLE 3. $\mathbb{T}_{k\geq 1} \equiv \mathbb{T}_{k+10} \pmod{30} = r_m \binom{d}{k'}^n$

2. DEFINING THE GENERIC OPERATOR λ

Definition 5 (Prerequisites to λ: recurrence relations in P)**.** Table 4 (below) illustrates the general observation between all series T'_k (including $\mathbb{T} = T'_1$, recursively) that some strata ν —which we shall designate as the generic operator in our precursor to λ— equal to one unit of generic k-prime-fields $\kappa_i = \{z : \kappa_n\}$, differences $\Delta_i = \{z : \Delta_n\}$, and constituent terms $\alpha_i = \{z : \alpha_n\}$ per series T'_k such that

$$\nu_k = \begin{cases} \kappa_n & \forall\, n \\ \Delta_n & \forall\, n\ . \\ \alpha_n & \forall\, n \end{cases}$$

We may define the operation by the recurrence relation

$$\nu(T'_k) = \Delta_\nu - \Delta_{\nu-1} + \alpha_{\nu-1} = \alpha_\nu\ ,\ k \geq 2. \tag{2.1}$$

$$\nu \begin{bmatrix} \{T'_k\} \\ \kappa_1 \\ \Delta_1 \\ \alpha_1 \\ \kappa_2 \\ \Delta_2 \\ \alpha_2 \\ \kappa_3 \\ \Delta_3 \\ \alpha_3 \\ \kappa_4 \\ \Delta_4 \\ \alpha_4 \\ \kappa_5 \\ \Delta_5 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} \{z\} & a & b & c & d & e & f & g & h & i & j \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 0 & 0 & 3 & 4 & 7 & 8 & 11 & 12 & 15 & 16 & 19 \\ 0 & 1 & 5 & 7 & 11 & 13 & 17 & 19 & 23 & 25 & 29 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 0 & 24 & 33 & 52 & 61 & 80 & 89 & 108 & 117 & 136 & 145 \\ 0 & 25 & 35 & 55 & 65 & 85 & 95 & 115 & 125 & 145 & 155 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 0 & 48 & 75 & 88 & 115 & 128 & 155 & 168 & 195 & 208 & 235 \\ 0 & 49 & 77 & 91 & 119 & 133 & 161 & 175 & 203 & 217 & 245 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 0 & 120 & 141 & 184 & 205 & 248 & 269 & 312 & 333 & 376 & 397 \\ 0 & 121 & 142 & 187 & 209 & 253 & 275 & 319 & 341 & 385 & 407 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 0 & 168 & 219 & 244 & 295 & 320 & 371 & 396 & 447 & 432 & 523 \\ 0 & 169 & 221 & 447 & 299 & 325 & 377 & 403 & 455 & 491 & 533 \end{bmatrix}$$

TABLE 4. $\Delta_\nu - \Delta_{\nu-1} + \alpha_{\nu-1} = \alpha_\nu$

Corollary (Calculating the real part of arbitrary α_i)**.** *Any term k' in some series T'_k where k'_j is the value of $\kappa_{\{z\}}$ for some rate $\rho : x$ (which varies as k of T'_k increases), it can be shown that*

$$\alpha_i = T'_{k'} = \sum_{i=k'}^{k' \to \infty} \kappa_i + \begin{cases} 4x & if\ k' > 0 \equiv 1 \\ 4x - 1 & if\ k' > 0 \equiv 0 \end{cases} \pmod 2.$$

Problem. A little brute force computation shows (per a given interval $300x$ per our $\pmod{30}$ array) the number of k terms in P that will be sifted by λ if constructable, shall vary as x increases. For $x = 1$, $\lambda_1 = 4$ and comprises the set $\{125, 175, 245, 275\}$; for $x = 2$, $\lambda_2 = 9$ and comprises the set $\{325, 343, 385, 425, 455, 475, 539, 575, 595\}$. How may we determine the terms of λ_3, and will it continue to follow this pattern? For some arbitrarily high value λ_n? Moreover, at what point shall intervals comprised of several units $300x$ occur, such that per any unit therein, all the possible locations $\pmod{30} \pm \{1, 5, 7, 11, 13\}$ may periodically be occupied by composites, or else elements isolated by λ?

Lemma. *The n-th term T'_n in any k-th series T'_k is equal to $2T'_{n-2} - T'_{n-4}$, $n \geq 5$.*
Remark. As was argued in the earlier Conjecture of this paper (and is easily demonstrated) the difference between any two consecutive terms in any series T'_k constitutes a binary operation: the difference between the second and first term will either be quadruple the input value when $k \equiv 1 \pmod 2$ (e.g. $5-1 = 1 \times 4$ for the recursive series T'_1; $77-49 = 7 \times 4$ for the series T'_3; etc.) or else the difference will be double the input value (e.g. $35-25 = 5 \times 2$ for the series T'_2; $143-121 = 11 \times 2$ for the series T'_4, etc.) if $k \equiv 0 \pmod 2$. The difference between the third and second term will be the inverse (double, if previously quadruple, and vice versa), then reverting to the initial difference for the fourth and third term, etc.
Conjecture. *Two classes of numbers encompass all term sifted by the λ-sieve operation, namely: 1) the second order product $\prod$ of any series T'_k a la equation (1.1) with each term multipled a second time by the correlated constant value P_k (e.g. the infinite series $T'_2 = 25_1, 35_2, 55_3, \cdots \pi_n$ multiplied by 5, yielding $\{125, 175, 275, 325, 375, 475, 525, 625, \cdots 5\pi_n\}$); as well as 2) all products of any terms $p^{n>1}$ and q^n where p and q are both contained in the set P and $p < q$.*

$$ds\begin{bmatrix} N\{z\} \\ 0\{z\} \\ 1\{z\} \\ 2\{z\} \\ 3\{z\} \\ 4\{z\} \\ 5\{z\} \\ 6\{z\} \\ 7\{z\} \\ 8\{z\} \\ 9\{z\} \end{bmatrix} = \begin{bmatrix} \{z\} & a & b & c & d & e & f & g & h & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 \\ 0 & 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 & 0 \\ 0 & 3 & 6 & 0 & 3 & 6 & 0 & 3 & 6 & 0 \\ 0 & 4 & 8 & 3 & 7 & 2 & 6 & 1 & 5 & 0 \\ 0 & 5 & 1 & 6 & 2 & 7 & 3 & 8 & 4 & 0 \\ 0 & 6 & 3 & 0 & 6 & 3 & 0 & 6 & 3 & 0 \\ 0 & 7 & 5 & 3 & 1 & 8 & 6 & 4 & 2 & 0 \\ 0 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = N_k \equiv N_{k\pm 9} \pmod 9$$

TABLE 5. $ds(\prod_{z=0}^{z=\infty} N\{z\}) \ : \ R = 10$

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